

# Critical Percolation on a Bethe Lattice Revisited\*

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**Abstract.** We introduce the model of independent percolation on general graphs with emphasis on the Bethe lattice, for which we prove the existence of a phase transition with the precise calculation of the critical value, and derive the value of the critical exponents  $\gamma$  and  $\beta$ .

**Key words.** Bethe lattice, critical exponent, percolation probability, asymptotic behavior

**AMS subject classifications.** 60A10, 05C90, 60K35

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**I. Introduction.** This paper is an attempt to introduce to undergraduate students in the sciences and engineering some concepts from the theory of percolation in a mathematically consistent way. A background in advanced calculus and probability at the level of [6] and [9], respectively, is desirable but not mandatory. The paper is self-contained and written in a “do it yourself” style, with many exercises left to the reader. Most of the available texts on the subject can be divided into two classes: those mathematically rigorous [16, 12, 8, 7, 18] but oriented to graduate students; and those physically appealing [20, 10] but lacking the mathematical rigor needed to develop a deeper understanding of the theory. We expect this paper to be useful, at the undergraduate level, to those approaching the subject from either point of view. The general aspects of the theory will be developed on *graphs* (see sections 2 and 3), but computations will be restricted to the *Bethe lattice* (see sections 4 to 7), where calculations can be handled. This is in contrast to more general graphs where, usually, no explicit calculations are possible (see section 3 and exercises therein), although in recent decades much effort has gone into understanding the theory on general graphs. After reading the paper, we expect readers to understand the meaning of *critical probability* and *critical exponents*, among other concepts.

Percolation theory was introduced in the early 1950s by Broadbent and Hammerley [5] and has attracted a lot of attention since then. Among its many applications, the most popular one is probably the flow of a fluid through a porous medium: Suppose that the pores of the medium are connected through channels and that each channel may be randomly open or closed to the passage of the flow, independently of the other. Equivalently, a channel is open with probability  $p$  and closed with probability  $1 - p$ ,  $0 \leq p \leq 1$ . We submerge the medium into a fluid and ask about the

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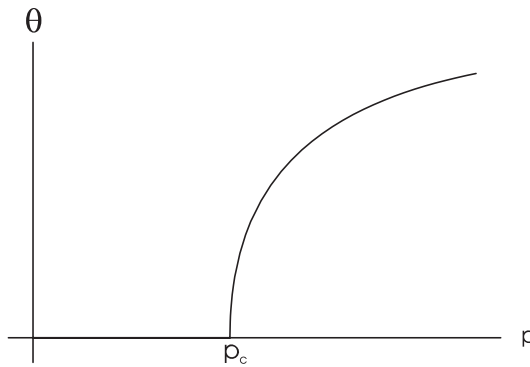


Fig. 1  $\theta$  versus  $p$ .

probability that the fluid reaches the center. Denote this probability by  $\theta$ . We say that the fluid *percolates* if the probability  $\theta$  is nonzero. It is intuitively clear that the probability  $\theta$  is a nondecreasing function of  $p$  and that  $\theta(0) = 0$  and  $\theta(1) = 1$ . The dependence of  $\theta$  on  $p$  can be measured experimentally, and it looks like Figure 1.

It is apparent that there is a distinguished point,  $p_c$ , where  $\theta$  changes from zero to a nonzero value. We say that there is a *phase transition* at  $p = p_c$  characterized by the change from zero to nonzero values of  $\theta$  and that  $p_c$  is the *critical probability*. Figure 1 suggests that there are examples in which the critical probability is nontrivial, i.e.,  $0 < p_c < 1$ . We will study such an example in section 3, where we define the function  $\theta$  properly, give a mathematical definition of  $p_c$ , and show the existence of a phase transition for models on Bethe lattices (see Definition 2.5 of a Bethe lattice). In section 5, we will find an expression for  $p_c$ . The behavior of the model around  $p = p_c$  is what we call *critical percolation*. Figure 1 suggests that  $\theta$  is continuous at  $p_c$ , i.e., that  $\theta$  approaches zero as  $p$  approaches  $p_c$  from above. When the approach is of the form

$$(1) \quad \theta(p) \approx (p - p_c)^\beta \quad \text{as } p \approx p_c, \quad p > p_c,$$

we say that  $\beta$  is a *critical exponent*. In section 6 we prove some properties of  $\theta$ , and based on them, in section 7 we prove the existence of the exponent  $\beta$  and that  $\beta = 1$  for any Bethe lattice.

**2. Graphs and Bethe Lattices.** In this section we introduce the notion of a graph and give some definitions which will be useful later on. The Bethe lattice is a special graph and is introduced in this section. See [4] for further reading.

**DEFINITION 2.1.** A graph  $G$  is a pair  $(V, \mathbb{E})$ , where  $V$  is a set of points, which we call vertices, and  $\mathbb{E}$  is a set of nonordered pairs of distinct vertices, which we call edges.

An edge  $e \in \mathbb{E}$  is denoted by  $\langle x, y \rangle$ , where  $x$  and  $y$  are the endpoints. A subgraph  $G'$  of a graph  $G = (V, \mathbb{E})$  is a pair  $(V', \mathbb{E}')$ , where  $V' \subset V$ ,  $\mathbb{E}' \subset \mathbb{E}$ , and the endpoints of edges in  $\mathbb{E}'$  are in  $V'$ . The *coordination number* of a vertex  $x$  is the number of edges that have  $x$  as an endpoint. A graph can be embedded in  $\mathbb{R}^3$ , but we may represent it in  $\mathbb{R}^2$  if we allow crossing of edges (see Figure 2).

**EXERCISE 2.1.** Write down the sets  $V$  and  $\mathbb{E}$  for the graph in Figure 2.

**EXERCISE 2.2.** Convince yourself that the pair  $(\mathbb{Z}^d, B_d)$ , where  $\mathbb{Z}^d$  is the set of  $d$ -dimensional vectors with integer entries and where  $B_d$  is the set of edges  $\langle x, y \rangle$  such

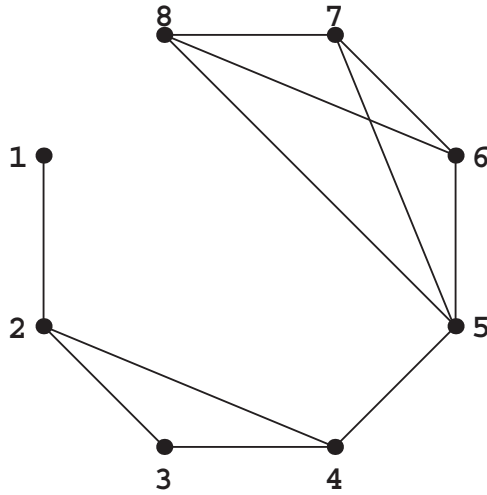


Fig. 2 A two-dimensional representation of a graph  $G$ .

that  $\sum_i |x_i - y_i| = 1$ , is a graph. Can you give other examples of graphs having  $\mathbb{Z}^d$  as vertices?

*Remark.* It is usual to represent the graph  $(\mathbb{Z}^d, B_d)$  just by  $\mathbb{Z}^d$ . This is known as the  $d$ -dimensional lattice. A vertex of  $\mathbb{Z}^d$  is usually called a *site*, while an edge is a *bond*.

A *path* is a finite set of edges  $\{e_1, e_2, \dots, e_n\}$ ,  $e_i = \langle x_i, x_{i+1} \rangle$  such that all vertices  $x_1, \dots, x_{n+1}$  are distinct from one another (see Figure 6(a)). We say that the path *begins* at  $x_1$  and *ends* at  $x_{n+1}$ . A *circuit* is a path  $\{e_1, e_2, \dots, e_n\}$  such that  $x_1 = x_{n+1}$ . A graph is *acyclic* if it has no circuits. We say that two vertices  $z$  and  $w$  are *connected* if there is a path  $\{e_1, e_2, \dots, e_n\}$  such that  $x_1 = z$  and  $x_{n+1} = w$  for some  $0 \leq i, j \leq n+1$ . A graph is finite if it has finitely many vertices and edges; it is infinite otherwise.

DEFINITION 2.2. A graph  $G$  is said to be connected if any two of its vertices are connected.

EXERCISE 2.3. Regarding the graph  $G$  of Figure 2, how many distinct paths connect vertices 1 and 5? How many circuits start and end at vertex 8? What is the coordination number of the vertices 1, 2, and 3?  $G$  is finite. Can you give an example of an infinite graph?

DEFINITION 2.3. A graph isomorphism from a graph  $G$  to a graph  $H$  is a bijection

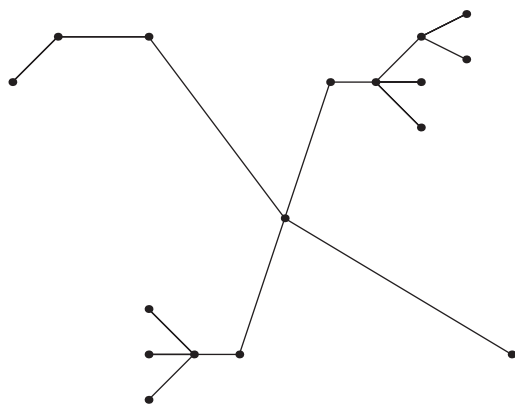
$$\sigma : V_G \rightarrow V_H$$

such that  $\langle v_1, v_2 \rangle \in \mathbb{E}_G$  if and only if  $\langle \sigma(v_1), \sigma(v_2) \rangle \in \mathbb{E}_H$ . Two graphs  $G$  and  $H$  are said to be isomorphic if there exists a graph isomorphism from  $G$  to  $H$ .

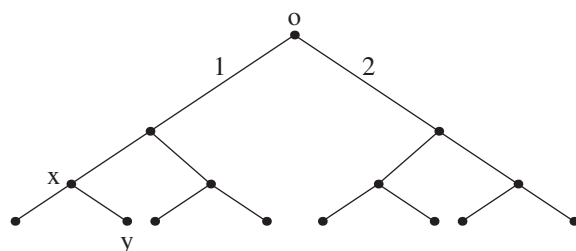
EXERCISE 2.4. Prove that if  $\sigma$  is a graph isomorphism from  $G$  to  $H$ , then the vertex bijection  $\sigma : V_G \rightarrow V_H$  preserves the coordination number of each vertex.

DEFINITION 2.4. A tree is a connected graph with the property that there exists a unique path connecting any two of its vertices (see Figure 3).

EXERCISE 2.5. Show that a graph is connected and acyclic if and only if it is a tree.




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**Fig. 3** A tree.



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**Fig. 4** Levels on the Bethe lattice  $\mathcal{B}_3$ .

The distance between two vertices of a graph is the *minimum* number of edges needed to connect them by a path. We denote the distance between vertices  $x$  and  $y$  by  $|x - y|$ .

**EXERCISE 2.6.** Show that, on a tree, the distance between two vertices  $x$  and  $y$  is exactly the number of edges in the unique path connecting  $x$  and  $y$ .

**DEFINITION 2.5.** A Bethe lattice is an infinite tree where each vertex has the same coordination number  $r$  except for one whose coordination number is  $r - 1$ . This particular vertex is the root or origin of the lattice.

The Bethe lattice was introduced by H. A. Bethe [3] in 1935 within the context of statistical mechanics. Since then it has been used as an approximation for models in high dimensions. We denote by  $\mathcal{B}_r$  the Bethe lattice with coordination number  $r$  (see Figure 4 for  $\mathcal{B}_3$ ). On  $\mathcal{B}_r$ , the distance of any vertex  $x$  to the origin is also known as the *level* of the vertex and is denoted by  $l(x)$ . For instance, in Figure 4,  $l(x) = 2$ , while  $l(y) = 3$ .

**EXERCISE 2.7.** Show that the positive one-dimensional lattice  $\mathbb{Z}^+$  is isomorphic to the Bethe lattice  $\mathcal{B}_2$ .

It is usual to consider a subgraph  $\mathcal{B}_r(L)$  of  $\mathcal{B}_r$ , made of vertices  $x$  such that  $l(x) \leq L$  and of edges connecting such vertices.  $\mathcal{B}_r(L)$  is said to be a “finite volume” Bethe lattice. The *boundary* of  $\mathcal{B}_r(L)$  is the set of vertices at level  $L$ , while the *surface area*  $S_r(L)$  is the number of vertices in the boundary. The *volume*  $V_r(L)$  of  $\mathcal{B}_r(L)$  is the number of vertices in  $\mathcal{B}_r(L)$ .

EXERCISE 2.8. Show by induction that the number of vertices at the  $k$ th level of a Bethe lattice is exactly  $(r - 1)^k$ ,  $k = 0, 1, \dots, L$ . In particular, conclude that  $S_r(L) = (r - 1)^L$ .

EXERCISE 2.9. Show that

$$V_r(L) = \frac{(r - 1)^{L+1} - 1}{r - 2}.$$

EXERCISE 2.10. Conclude from the above exercises that

$$\lim_{L \rightarrow \infty} \frac{S_r(L)}{V_r(L)} = \frac{(r - 2)}{(r - 1)}.$$

Therefore, we conclude from Exercise 2.10 that if  $r > 2$ , then the surface area and the volume of a finite volume Bethe lattice grow at the same rate as  $L \rightarrow \infty$ . In  $\mathbb{Z}^d$  or  $\mathbb{R}^d$  one would expect the surface area to grow more slowly than the volume. For instance, let  $\Lambda_L = \{x \in \mathbb{Z}^d; |x_i| \leq L\}$  be a finite box in  $\mathbb{Z}^d$  and let  $\partial\Lambda_L = \{x \in \mathbb{Z}^d; |x_i| = L \text{ for some } i = 1, \dots, d\}$  be its boundary. The surface of  $\Lambda_L$  is  $S(L) = 2d(2L)^{d-1}$  and its volume is  $V(L) = L^d$ , so that  $[S(L)/V(L)] = (2d/L) \rightarrow 0$  as  $L \rightarrow \infty$ . On the other hand, if we allow  $L$  to depend on the space dimension  $d$ , as, for instance,  $L = \alpha d$ , and if we let  $d \rightarrow \infty$ , then, again for the  $d$ -dimensional cube,  $[S(L)/V(L)] \rightarrow (2/\alpha)$ . Therefore, models on a Bethe lattice are viewed as approximations for behavior in very high space dimensions.

**3. Percolation on Graphs.** In this section we will define the percolation model on an arbitrary connected graph  $G$ . First we define the percolation model on a finite graph. In particular, we define what we mean by *occurrence of percolation* on a finite box of  $\mathbb{Z}^d$ . Then, we extend the percolation model to an infinite graph.

Consider an arbitrary finite connected graph  $G$ . Let  $0 < p < 1$ . Suppose that you can close or let open the edges of this graph with probability  $p$  or  $1 - p$ , respectively. So, we introduce the *random variables*  $\omega_e$ ,  $e \in \mathbb{E}$ , that assume the value 1 with probability  $p$  (meaning that the edge  $e$  remains open) or the value 0 with probability  $1 - p$  (meaning that the edge  $e$  becomes closed). We assume that the states of the edges are *independent* of each other. In particular,  $\{\omega_e\}_{e \in \mathbb{E}}$  is a collection of independent, identically distributed Bernoulli- $p$  random variables. In this way we obtain the *independent percolation model* on the edges<sup>1</sup> of the graph  $G$ , called the *bond percolation model*.

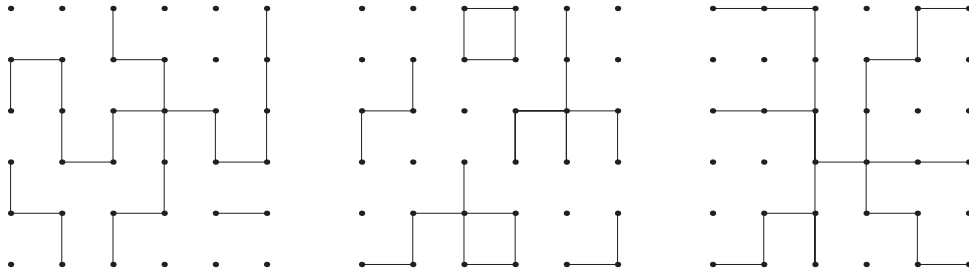
On a finite graph, a *configuration*  $\omega$  of this model can be seen as a vector with as many entries as the number of edges of  $G$ , each entry being 0 or 1. More specifically, a configuration  $\omega$  is a point of the set  $\Omega = \{0, 1\}^{\mathbb{E}}$ , the Cartesian product of  $\{0, 1\}$  with itself as many times as the number of edges of  $G$ .  $\Omega$  is known as the *space of configurations*. On this graph  $G$  we can define the probability of occurrence of a configuration  $\omega$ :

$$(2) \quad P_p(\omega) = p^{|O_\omega|} (1 - p)^{|F_\omega|},$$

where  $|O_\omega|$  and  $|F_\omega|$  are the number of open and closed edges of  $\omega$ , respectively.

EXERCISE 3.1. Consider a percolation model defined on a  $6 \times 6$  two-dimensional lattice. Open edges are represented by line segments. In terms of  $p$ , compute the probability of the occurrence of each one of the configurations given in Figure 5.

<sup>1</sup>One can also define an independent percolation model on the vertices of  $G$ , instead of edges. We just have to associate a random variable  $\omega_x$  with each vertex  $x$ , which assumes the value 1 with probability  $p$  and the value 0 with probability  $1 - p$ . This is known as *site percolation*.



**Fig. 5** Three distinct configurations on a  $6 \times 6$  lattice.

More generally, one can define the probability of subsets of  $\Omega$ . Let  $A \subset \Omega$  be the set of configurations  $A = \{\omega_1, \dots, \omega_n\}$ . Then the probability of occurrence of  $A$  is

$$(3) \quad P_p(A) = \sum_{i=1}^n P_p(\omega_i).$$

In particular, one has the following:

1.  $P_p(\emptyset) = 0$  and  $P_p(\Omega) = 1$ ;
2. for any collection  $A_1, \dots, A_k$  of pairwise disjoint subsets of  $\Omega$ ,

$$(4) \quad P_p\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k P_p(A_i).$$

EXERCISE 3.2. Use the binomial theorem to prove that  $P_p(\Omega) = 1$ .

We can now define what is meant by *occurrence of percolation* in the context of finite boxes<sup>2</sup>  $\Lambda_L$  of  $\mathbb{Z}^d$ . Let  $\Theta_L$  be the set of configurations

$$(5) \quad \Theta_L = \{\omega \in \Omega; 0 \text{ is connected to a site in } \partial\Lambda_L\}.$$

DEFINITION 3.1. We say that percolation occurs in  $\Lambda_L$  and at probability  $p$  if  $P_p(\Theta_L) > 0$ .

One can check from (2) that if  $G$  is a finite graph and if  $0 < p < 1$ , then any configuration has a nonzero probability of occurrence. One can also check that, for any finite box  $\Lambda_L$ ,  $\Theta_L$  is a nonempty set. It follows from these remarks that  $P_p(\Theta_L) > 0$  for any finite box and for any  $p \in (0, 1]$ . In particular, the probability that percolation occurs is always nonzero, suggesting that  $p_c$  (see Figure 1) is zero (since  $\theta = 0$  if  $p = 0$  and  $\theta > 0$  if  $p > 0$ ). This shows that a percolation model on a finite graph is not suitable to reproduce the critical phenomenon mentioned in the introduction. Therefore, in order to get a nonzero value for  $p_c$ , we must consider models defined on infinite graphs. We must be careful, though, since the maximal probability of a configuration on a graph of  $k$  edges is equal to  $(\bar{p})^k$ , where  $\bar{p} = \max\{p, 1 - p\}$ , and it goes to zero when the number of edges grows to infinity, making it impossible to assign a nonzero probability value to a single configuration on an infinite graph. We are led to define the probability of more general sets of configurations.

A subset of  $\Omega$  that contains all of the configurations with specified states on a finite number of edges is known as a *cylinder set*. Any cylinder set is of the form  $\{\omega \in \Omega :$

<sup>2</sup>We observe that the same definition holds for finite Bethe lattices  $\mathcal{B}_r(L)$ .

$\omega_{e_i} = \sigma_{e_i} \forall i = 1, \dots, n \} \equiv \mathcal{C}(\{\sigma_{e_1}, \dots, \sigma_{e_n}\})$ , where  $n$  is a positive natural number,  $\sigma_{e_i} = 0$  or  $1$  for  $i = 1, \dots, n$ , and the values  $\sigma_{e_1}, \dots, \sigma_{e_n}$  are given beforehand. Due to the independence among the states of the edges, it is easy to compute the probability of a cylinder set. Let  $|O_\sigma|$  and  $|F_\sigma|$  be the number of open and closed edges of  $\{\sigma_{e_1}, \dots, \sigma_{e_n}\}$ , respectively. Equivalently,  $|O_\sigma|$  is the number of  $\sigma$ 's equal to  $1$  and  $|F_\sigma|$  is the number of  $\sigma$ 's equal to  $0$ . Then

$$(6) \quad P_p(\mathcal{C}(\{\sigma_{e_1}, \dots, \sigma_{e_n}\})) = p^{|O_\sigma|}(1-p)^{|F_\sigma|}.$$

Observe that the above formula gives the probability for the occurrence of a *set of configurations* on the graph  $G$ , whether or not the graph is finite. In particular, if  $G$  is a finite graph having  $n$  edges, then  $\mathcal{C}(\{\sigma_1, \sigma_2, \dots, \sigma_n\})$  contains a single configuration and (6) reduces to (2).

EXERCISE 3.3. *Show that the finite intersection of cylinder sets is a cylinder set.*

To proceed with our discussion, we should define what the occurrence of percolation means in the context of an infinite graph. Let  $x$  be a vertex of  $G$ . The (random) set of  $G$  that contains all vertices which are connected to  $x$  by a path of open edges is called the *open cluster* at  $x$  and we denote it by  $C(x)$ . The number of vertices of  $C(x)$  is denoted by  $|C(x)|$ . We also denote by  $C$  the open cluster at the origin (see Figure 6(b)). In analogy with the definition of  $\Theta_L$  (see (5)), we define  $\Theta$  to be the set

$$\Theta = \{\omega \in \Omega; |C| = \infty\}.$$

EXERCISE 3.4. *Explain why  $\Theta$  is not a cylinder set.*

For infinite graphs, we want to extend the definition of  $P_p$  to include more subsets of  $\Omega$  than just the cylinder sets. In particular, we wish to assign a probability to  $\Theta$  (which, by Exercise 3.4, is not a cylinder set).

Due to Exercise 3.3, we can define, for any finite collection of cylinder sets  $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ , the probability of occurrence of their union:

$$(7) \quad P_p(\cup_{i=1}^n \mathcal{C}_i) = \sum_{1 \leq i \leq n} P_p(\mathcal{C}_i) - \sum_{1 \leq i < j \leq n} P_p(\mathcal{C}_i \cap \mathcal{C}_j) + \dots + (-1)^{n+1} P_p(\mathcal{C}_1 \cap \dots \cap \mathcal{C}_n),$$

which is just the *inclusion-exclusion* formula. We notice that, as for finite graphs,  $P_p(\emptyset) = 0$ ,  $P_p(\Omega) = 1$ , and (4) remains valid for a disjoint union of cylinder sets.

Let  $\mathcal{A}$  be the family of all possible finite unions of cylinder sets. By (7), a probability can be assigned to all sets in  $\mathcal{A}$ .

EXERCISE 3.5. *Show that  $\mathcal{A}$  is an algebra of sets, i.e.,*

1.  $\emptyset, \Omega \in \mathcal{A}$ ;
2. if  $A \in \mathcal{A}$ , then the complement  $A^c \in \mathcal{A}$ ;
3. if  $A_1, A_2, \dots, A_n \in \mathcal{A}$ , then  $\bigcup_{k=1}^n A_k \in \mathcal{A}$ .

$\mathcal{A}$  is known as the *algebra generated by the cylinder sets*. But the algebra  $\mathcal{A}$  is not satisfactory yet, as we can see by solving the next exercise.

EXERCISE 3.6. *Explain why, on an infinite graph,  $\Theta$  is not in  $\mathcal{A}$ .*

Since our goal is to assign a probability of occurrence to  $\Theta$ , we need to extend  $P_p$  to an even larger collection of subsets of  $\Omega$ . For this we rely on some rather highly technical mathematical machinery, the Carathéodory theorem, which can be used to extend  $P_p$  from the algebra  $\mathcal{A}$  to the so-called  $\sigma$ -algebra generated by  $\mathcal{A}$ .

DEFINITION 3.2. A family  $\mathcal{F}$  of subsets of a set  $\Omega$  is said to be a  $\sigma$ -algebra if conditions 1 and 2 of Exercise 3.5 are satisfied and if  $(F_n)$  is a sequence of sets in  $\mathcal{F}$ , then  $\bigcup_{n=1}^{\infty} F_n \in \mathcal{F}$ .

EXERCISE 3.7. Prove that if  $(F_n)$  is a sequence of sets in the  $\sigma$ -algebra  $\mathcal{F}$ , then  $\bigcap_{n=1}^{\infty} F_n \in \mathcal{F}$ .

Given the family  $\mathcal{A}$  of subsets of  $\Omega$ , we can generate the smallest  $\sigma$ -algebra containing  $\mathcal{A}$  by taking the intersections of all  $\sigma$ -algebras containing  $\mathcal{A}$ . This smallest  $\sigma$ -algebra is often called the  $\sigma$ -algebra generated by  $\mathcal{A}$  and it will contain, in particular, all countable unions of sets in  $\mathcal{A}$ .

EXERCISE 3.8 (challenging exercise). Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by the cylinder sets on  $\mathbb{Z}^d$ . Consider the sets  $\Theta_n = \{0 \text{ is connected to the boundary of } \Lambda_n\}$ ,  $n = 1, 2, \dots$ . By a limit procedure, show that  $\Theta$  is in  $\mathcal{F}$ .

DEFINITION 3.3. A nonnegative set function  $P(\cdot)$  defined on a  $\sigma$ -algebra  $\mathcal{F}$  of subsets of  $\Omega$  is a probability measure (or a probability) if

1.  $P(\Omega) = 1$ ;
2. for every pairwise disjoint countable collection  $\{F_k\}$  of sets in  $\mathcal{F}$ ,

$$(8) \quad P\left(\bigcup_k F_k\right) = \sum_k P(F_k).$$

Remark. Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by the algebra  $\mathcal{A}$  of cylinder sets. We have already seen that (6) can be extended to a probability  $P_p(\cdot)$  on  $\mathcal{A}$ . Then the Carathéodory extension theorem (see [19, p. 257]) allows us to extend  $P_p(\cdot)$  to a unique probability measure on  $\mathcal{F}$ .

The probability  $P_p(\cdot)$  is known as the *percolation process*. In measure theory language, we have generated a *probability space* that consists of a triple  $(\Omega, \mathcal{F}, P_p)$  and, in the rest of the paper, we will work on this space.

EXERCISE 3.9. Let  $A, B \in \mathcal{F}$ . Using (8), prove that  $P_p(A) \leq P_p(B)$  if  $A \subset B$ .

EXERCISE 3.10. Let  $A_1, A_2 \in \mathcal{F}$ . Prove that  $P_p(A_1 \cup A_2) \leq P_p(A_1) + P_p(A_2)$ . Generalize this exercise to  $A_1, A_2, \dots, A_n \in \mathcal{F}$ .

DEFINITION 3.4. An element of  $\mathcal{F}$  is an event. Two events are said to be independent if their occurrence depends on disjoint sets of edges.

If  $A$  and  $B$  are independent, then it is possible to prove that

$$(9) \quad P_p(A \cap B) = P_p(A) \cdot P_p(B).$$

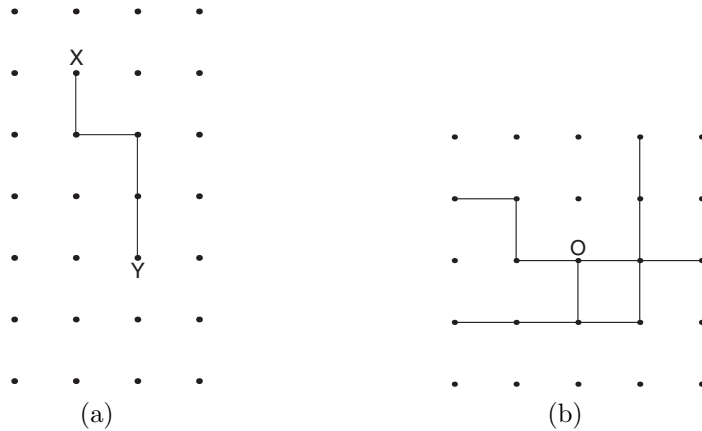
We can also define the expected value with respect to the probability measure  $P_p$  of an  $\omega$ -dependent function. For instance, fix a finite set of edges  $E = \{e_1, \dots, e_k\}$  and suppose that  $f(\omega)$  depends only on the edges in  $E$ . Then the *expected value*, or the *integral*, of  $f$  is defined as follows:

$$(10) \quad E_p(f) = \sum_{\{\sigma_1, \dots, \sigma_k : \sigma_i = 0, 1\}} f(\sigma_1, \dots, \sigma_k) P_p(\omega \in \Omega : \omega_i = \sigma_i \forall i = 1, \dots, k).$$

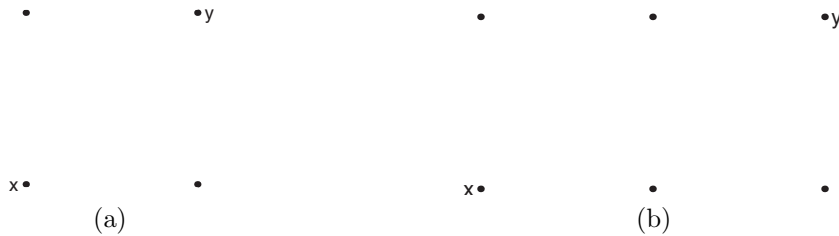
By a limiting procedure, the notion of expected value can be extended to functions  $g$  that depend on an infinite number of variables. The procedure reduces to approximating  $g$  by functions of many variables and then taking limits. We will assume that this has been done (but see [19, p. 225]).

EXERCISE 3.11. Let  $f$  be a function that depends only on  $\omega_e$ , for a fixed edge  $e$ . Compute its expected value as a function of  $p$ . Do the same exercise for a function depending only on  $\omega_{e_1}$  and  $\omega_{e_2}$ , for fixed edges  $e_1$  and  $e_2$ .





**Fig. 6** (a) An open path connecting  $x$  and  $y$  for a given configuration. (b) The open cluster at the origin for a given configuration.



**Fig. 7** (a) A  $2 \times 2$  lattice. (b) A  $2 \times 3$  lattice.

We end this section with some exercises which will be useful in what follows.

**EXERCISE 3.12.** Let  $\Gamma$  be a path on a graph  $G$  where a bond percolation model has been defined. Show that  $\{\omega \in \Omega : \Gamma \text{ is open}\}$  is a cylinder set and compute its probability.

**EXERCISE 3.13.** Let  $G$  be a graph with an countable set of edges and let  $\{x \longleftrightarrow y\} \equiv \{\omega \in \Omega : \text{there exists an open path connecting } x \text{ and } y\}$ . Show that this set is an event (see Figure 6(a)). Is it a cylinder set?

**EXERCISE 3.14.** Given any natural number  $n \geq 1$ , show that  $\{\omega \in \Omega : |C| = \infty\} \subset \{\omega \in \Omega : |C| \geq n\}$ .

**EXERCISE 3.15.** Given the lattice in Figure 7(a), compute the probability that  $x$  will be connected to  $y$ . Do the same exercise for the lattice in Figure 7(b).

**4. Computation of Probabilities on a Bethe Lattice.** From now on we will restrict ourselves to the Bethe lattice  $\mathcal{B}_r$ . We recall that the computation of the probability in Exercise 3.15 gets harder as we increase the size of the lattice because the number of open paths connecting the origin  $O$  to  $x$  increases as the size of the lattice increases. Working on a Bethe lattice avoids this complication, allowing us to compute many interesting quantities explicitly. We point out that the study of the percolation on a Bethe lattice was started by Flory [11] in 1941.

The *connectivity function*, denoted by  $\tau_{xy}(p)$ , measures the probability that the vertices  $x$  and  $y$  will be connected by an open path, i.e.,  $\tau_{xy}(p) = P_p(\{x \longleftrightarrow y\})$ .

PROPOSITION 4.1. On  $\mathcal{B}_r$ ,

$$(11) \quad \tau_{xy}(p) = p^{|x-y|}.$$

*Proof.* From the tree structure of  $\mathcal{B}_r$ , there is only one way to connect  $x$  to  $y$  by a path of edges starting at  $x$  and ending at  $y$ . Let us denote this path by  $\Gamma_{xy}$ . It is clear that

$$\{x \longleftrightarrow y\} = \{\Gamma_{xy} \text{ is open}\}.$$

Therefore,  $\tau_{xy}(p) = P_p(\{\Gamma_{xy} \text{ is open}\}) = p^{|x-y|}$ .  $\square$

EXERCISE 4.1. Try to compute  $\tau_{xy}(p)$ , where  $x$  and  $y$  are nearest neighbors, on the lattice  $\mathbb{Z}^2$ . Can you write it in closed form? Why not?

*Remark.* We have already seen in Exercise 2.10 that the Bethe lattice  $\mathcal{B}_r$  has a peculiar behavior as the infinite volume limit is taken. The percolation model on  $\mathcal{B}_r$  has some peculiarities too. For instance, the connectivity function on  $\mathcal{B}_r$  decays exponentially fast as  $|x-y| \rightarrow \infty$  for all  $p < 1$ . This can be seen from (11) since  $\tau_{xy}(p) = \exp(-|x-y|/\zeta)$ , with  $\zeta(p) \equiv (-\ln p)^{-1}$ . The function  $\zeta$  is usually referred to as the *correlation length*. Therefore, on a Bethe lattice, the correlation length is finite for all  $0 < p < 1$ , while, on  $\mathbb{Z}^d$ , the correlation length blows up as  $p$  approaches  $p_c$  from below.

The *percolation probability*, denoted by  $\theta(p)$ , measures the probability that the open cluster at the origin will be infinite, i.e.,  $\theta(p) = P_p(\{|C| = \infty\})$ .

PROPOSITION 4.2. On  $\mathcal{B}_r$ ,

$$(12) \quad \theta(p) = 0 \quad \forall p; \quad p < \frac{1}{r-1}.$$

*Proof.* For all  $n \geq 1$ ,

$$\{|C| = \infty\} \subset \cup_{\{x: l(x)=n\}} \{0 \longleftrightarrow x\}.$$

Then, from Proposition 4.1, Exercise 3.10, and the above inclusion,

$$(13) \quad \theta(p) = P_p(\{|C| = \infty\}) \leq \sum_{\{x: l(x)=n\}} P_p(\{0 \longleftrightarrow x\}) = (r-1)^n p^n,$$

where, in the last equality, we have used Exercise 2.8 to conclude that

$$\sum_{\{x: l(x)=n\}} 1 = (r-1)^n.$$

Assuming that  $p < (r-1)^{-1}$ , we take the limit  $n \rightarrow \infty$  and get the result.  $\square$

Proposition 4.2 allows us to define a *critical probability*, denoted by  $p_c$ , as being the largest value of  $p$  below which  $\theta(p) = 0$ :

$$(14) \quad p_c \equiv \sup\{p \geq 0 : \theta(p) = 0\}.$$

It is clear that  $p_c \leq 1$ . From (12) we have that  $p_c \geq (r-1)^{-1}$ . Therefore,  $0 < p_c \leq 1$  for any  $r \geq 2$ .

EXERCISE 4.2. Show that  $p_c = 1$  if  $r = 2$ .

The meaning of the above exercise is that, if  $r = 2$ , then the infinite cluster at the origin will show up only at  $p = 1$ . It is natural to ask whether there are values

of  $r$  for which  $p_c$  is less than 1. In section 5 we will show that  $p_c = (r - 1)^{-1}$ , so that  $0 < p_c < 1$  for any  $r \geq 3$ . The critical probability  $p_c$  and the function  $\theta$  play the same role that the *critical temperature* and the *magnetization*, respectively, play in magnetic systems. We say that there is a *phase transition*<sup>3</sup> at  $p_c$ . The function  $\theta$  is viewed as the *order parameter* for this transition:  $\theta = 0$  indicates the *subcritical region*  $p < p_c$  while  $\theta > 0$  indicates the *supercritical region*  $p > p_c$ . As suggested by Figure 1, we will show below that  $\theta(p_c) = 0$  (see Corollary 5.2) but, even so,  $p_c$  is not considered as a point in the subcritical region. Instead,  $p_c$  is a special point around which functions behave in a special way.

There are other possibilities for defining a critical probability. This can be done by introducing the function *expected cluster size* (also known as *susceptibility*),  $\chi(p) \equiv E_p(|C|)$ .

PROPOSITION 4.3.

$$(15) \quad \chi(p) = \sum_{x \in \mathcal{B}_r} \tau_{0x}(p).$$

*Proof.* Let  $\mathbf{1}_{0x}(\omega)$ ,  $\omega \in \Omega$ , denote the *characteristic function* of the event  $\{0 \longleftrightarrow x\}$ , i.e.,  $\mathbf{1}_{0x}(\omega) = 1$  if  $\omega \in \{0 \longleftrightarrow x\}$  and  $\mathbf{1}_{0x}(\omega) = 0$  otherwise. Then, for each  $\omega \in \Omega$ , the size of the open cluster at the origin can be written as

$$|C(\omega)| = \sum_{x \in \mathcal{B}_r} \mathbf{1}_{0x}(\omega).$$

Taking expected values on both sides and observing that  $E_p(\mathbf{1}_{0x}(\omega)) = \tau_{0x}(p)$ , we get the result.  $\square$

The susceptibility function  $\chi(p)$  may be explicitly computed, as the next proposition shows.

PROPOSITION 4.4. *On  $\mathcal{B}_r$ ,*

$$(16) \quad \chi(p) = \begin{cases} (1 - (r - 1)p)^{-1} & \text{if } p < \frac{1}{r-1}, \\ \infty & \text{otherwise.} \end{cases}$$

*Proof.* From Proposition 4.1 and from identity (15), we have that

$$\begin{aligned} \chi(p) &= \sum_{x \in \mathcal{B}_r} \tau_{0x}(p) \\ &= \sum_{x \in \mathcal{B}_r} p^{l(x)} = \sum_{l \geq 0} \sum_{\{x \in \mathcal{B}_r : l(x)=l\}} p^{l(x)} = \sum_{l \geq 0} (r - 1)^l p^l. \end{aligned}$$

Now, the result comes from the last sum since it is convergent if and only if  $p < (r - 1)^{-1}$ .  $\square$

Proposition 4.4 allows us to define another critical probability, which we will denote by  $\pi_c$ :

$$(17) \quad \pi_c \equiv \inf\{p : \chi(p) = \infty\}.$$

From the identity (16) and from definition (17) it is clear that  $\pi_c = (r - 1)^{-1}$ . Comparing the above definition with (14) we conclude that  $p_c \geq \pi_c$ . From the results

<sup>3</sup>We have already observed that, for the percolation model on  $\mathbb{Z}^d$ , the correlation length  $\zeta(p)$  blows up as we approach  $p_c$  from below. This behavior characterizes a *second-order* phase transition at  $p_c$ .

in the next section we will get the equality between the two definitions, i.e.,  $p_c = \pi_c$ . This result<sup>4</sup> is important because it does not allow for the possibility of an *intermediate phase* where  $\chi(p) = \infty$  but  $\theta(p) = 0$  for  $\pi_c < p < p_c$ .

From the representation formula (16) for  $\chi(p)$ ,  $p < (r-1)^{-1}$ , we conclude that  $\chi(p) \rightarrow \infty$  as  $p$  approaches  $p_c$  from the left. An interesting question is to find out at which rate  $\chi$  approaches  $\infty$ . If there is an exponent  $\gamma > 0$  such that the limit

$$\lim_{p \rightarrow p_c^-} (p_c - p)^\gamma \chi(p)$$

exists and is nonzero, then  $\gamma$  is said to be the *critical exponent* for  $\chi(p)$ .

EXERCISE 4.3. *In the next section we will prove that  $p_c = \pi_c$ . Using this fact, show that*

$$\chi(p) = \frac{p_c}{p_c - p}$$

for  $p < p_c$  and for any  $r \geq 2$ . Conclude that  $\gamma = 1$  for any  $r \geq 2$ .

*Remark.* We have already informally introduced the exponent  $\beta$ ; see (1). It is a harder task to prove that  $\beta = 1$ . This will be done in section 7.

**5.  $p_c = (r-1)^{-1}$  on the Bethe Lattice  $\mathcal{B}_r$ .** In this section we will show that  $\theta(p) > 0$  for all  $p > (r-1)^{-1}$ . Using the definition (14) of  $p_c$ , this result implies that  $p_c \leq (r-1)^{-1}$  and therefore that  $p_c = (r-1)^{-1}$  because we already know from Proposition 4.2 that  $p_c \geq (r-1)^{-1}$ .

Let  $C$  denote the open cluster at the origin of  $\mathcal{B}_r$ . Noting that  $r-1$  branches pop up from the origin, we denote by  $C_i$  the connected component of  $C$  that is fully contained in the  $i$ th branch (for  $r=3$  we have two branches (1 and 2); see Figure 4).

EXERCISE 5.1. *Show that*

$$\{|C| < \infty\} = \bigcap_{i=1}^{r-1} \{|C_i| < \infty\}$$

and conclude that

$$P_p(\{|C| < \infty\}) = P_p\left(\left\{\{|C_1| < \infty\} \cap \{|C_2| < \infty\} \cap \cdots \cap \{|C_{r-1}| < \infty\}\right\}\right).$$

EXERCISE 5.2. *Show that the events  $\{|C_i| < \infty\}$  and  $\{|C_j| < \infty\}$ ,  $i \neq j$ , are independent.*

Define  $Q_i \equiv P_p(\{|C_i| < \infty\})$  and observe that, by symmetry,  $Q_i = Q_j = Q$  for all  $i \neq j$ . Furthermore, since the events  $\{|C_i| < \infty\}$  and  $\{|C_j| < \infty\}$  are independent for all  $i \neq j$ , we have that

$$\begin{aligned} & P_p\left(\left\{\{|C_1| < \infty\} \cap \{|C_2| < \infty\} \cap \cdots \cap \{|C_{r-1}| < \infty\}\right\}\right) \\ (18) \quad & = \prod_{i=1}^{r-1} P_p(\{|C_i| < \infty\}) = Q^{r-1}. \end{aligned}$$

<sup>4</sup>It is a hard task to prove that  $\pi_c = p_c$  for other graphs. In the mid-1980s, Menshikov and Sidorenko [17] and Aizenman and Barsky [1] independently proved the result for translation-invariant lattices in general dimensions (in particular, they proved it for  $\mathbb{Z}^d$ ).

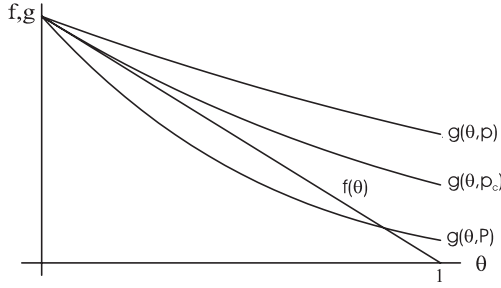


Fig. 8 Graphs of  $f$  and  $g$  for various values of  $p$ .

Since  $1 - \theta(p) = P_p(\{|C| < \infty\})$ , from (18) we have

$$(19) \quad 1 - \theta(p) = Q^{r-1}.$$

In order to compute  $Q$ , we fix a branch emerging from the origin, say, the leftmost one, denoted by  $C_1$ , and perform the calculations from this branch. We note that this branch consists of a distinguished edge  $e_1$  attached to a graph which is isomorphic to the original Bethe lattice. Let  $\tilde{C}_i$  be the connected component of this graph that is fully contained in the  $i$ th branch. Then

$$(20) \quad Q = P_p \left( \left\{ \{\omega_{e_1} = 0\} \cup \left\{ \{\omega_{e_1} = 1\} \cap \{|\tilde{C}_1| < \infty\} \cap \dots \cap \{|\tilde{C}_{r-1}| < \infty\} \right\} \right\} \right).$$

Using the facts that the events  $\{\omega_{e_1} = 0\}$  and  $\{\omega_{e_1} = 1\}$  are disjoint, that the events  $\{\omega_{e_1} = 1\}$  and  $\{|\tilde{C}_i| < \infty\}$ ,  $i = 1, \dots, r - 1$ , are independent, and that  $P_p(\{|C_i| < \infty\}) = Q = P_p(\{|\tilde{C}_j| < \infty\})$ , from (20) we get

$$(21) \quad Q = (1 - p) + pQ^{r-1}.$$

Replacing (19) in (21), we finally get

$$(22) \quad 1 - \theta = (1 - p\theta)^{r-1}.$$

Equation (22) is our starting point (see [20, 10]). It defines  $\theta$  implicitly as a function of  $p$ , for  $p > 0$ . Observe that  $\theta = 0$  is always a solution to (22). We will show below that (22) has a unique nonzero solution  $\theta$  if and only if  $p > (r - 1)^{-1}$ .

**THEOREM 5.1.** *Suppose that  $r \geq 3$ . Then (22) has a unique nonzero solution if and only if  $p > (r - 1)^{-1}$ .*

*Proof.* Let  $f(\theta) \equiv 1 - \theta$  and  $g(p, \theta) \equiv (1 - p\theta)^{r-1}$ . Equation (22) is satisfied if and only if  $f(\theta) = g(p, \theta)$ . In order to prove the assertion, we use a graphical method. We fix the value of  $p < 1$  and, by analyzing the graphs of  $f$  and  $g$  (see Figure 8) as functions of  $\theta$ , for  $\theta$  in the interval  $[0, 1]$ , we conclude that they cross at a unique nonzero value of  $\theta$  if and only if  $p > (r - 1)^{-1}$ .

First, note that the graph of  $f(\theta)$  is a straight line with slope  $-1$  and  $y$ -intercept 1. Now, since  $f(0) = g(p, 0) = 1$  for any  $p > 0$ , the graphs of  $f$  and  $g$ , taken as a function of  $\theta$  only, both start at the same value. On the other hand, at  $\theta = 1$ ,  $f$  is below  $g$  since  $f(1) = 0 < (1 - p)^{r-1} = g(1)$ . The  $\theta$ -derivative of  $g$  is equal to

$$\frac{\partial g(p, \theta)}{\partial \theta} = -p(r - 1)(1 - p\theta)^{r-2}.$$

It is negative for  $\theta \in [0, 1)$ . Therefore,  $g$  is a decreasing function of  $\theta$  and, at  $\theta = 0$ ,

$$(23) \quad \left[ \frac{\partial g}{\partial \theta} \right]_{\theta=0} \begin{cases} > -1 & \text{if } p < (r-1)^{-1}, \\ = -1 & \text{if } p = (r-1)^{-1}, \\ < -1 & \text{if } p > (r-1)^{-1}. \end{cases}$$

The second  $\theta$ -derivative of  $g$ , which is given by

$$(24) \quad \frac{\partial^2 g}{\partial \theta^2} = p^2(r-1)(r-2)(1-p\theta)^{r-2},$$

is positive, if  $r \geq 3$ , throughout the region we are interested in, so the first derivative of  $g$  is strictly increasing.

From the monotonicity of  $g$  and from the comparison between its  $\theta$ -derivative at the origin (see identity (23)) and  $-1 = f'(0)$ , we conclude that when  $p \leq (r-1)^{-1}$ , the graph of  $g$  is above the graph of  $f$  and the origin is the only point where the graph of  $g$  intercepts the graph of  $f$ . On the other hand, when  $p > (r-1)^{-1}$ , the graph of  $g$  is below the graph of  $f$ , at least for values of  $\theta$  close to zero. Since  $g(p, 1) = (1-p)^{r-1} \geq 0 = f(1)$  if  $p \leq 1$  and since  $g$  is a continuous function of  $\theta \in [0, 1)$ , its graph will necessarily cross the graph of  $f$  at a nonzero value of  $\theta$ , and the uniqueness of this point is given by (24) (since  $g$  is concave, its graph intercepts the graph of  $f$  in at most two distinct points), proving the proposition.  $\square$

COROLLARY 5.1. *On  $\mathcal{B}_r$ , if  $r \geq 3$ ,*

$$p_c = \frac{1}{r-1}.$$

*Remark.* It is harder to prove that  $p_c < 1$  for percolation on  $\mathbb{Z}^d$ . This is accomplished by using *Peierls' argument* to conclude that  $\theta(p) > 0$  for  $p$  close to 1 (see, for example, [12]).

COROLLARY 5.2.  $\theta(p_c) = 0$  on a Bethe lattice with coordination number  $r \geq 3$ .

EXERCISE 5.3. *Prove Corollaries 5.1 and 5.2 (see the beginning of this section and the proof of Theorem 5.1, taking  $p = p_c$ ).*

*Remark.* We point out that, on  $\mathbb{Z}^d$ , the result  $\theta(p_c) = 0$  has been established by Kesten [15] for  $d = 2$  and by Hara and Slade [13, 14] for  $d \geq 19$ . See also the remark at the end of this paper.

**6. Analytical Properties of  $\theta(p)$ .** In this section we will establish the continuity of  $\theta(p)$  in its whole interval of definition. Actually, it is enough to prove continuity only at  $p_c$ . Indeed, since  $\theta(p) = 0$  for  $0 \leq p < p_c$ , it is continuous in  $[0, p_c)$  (right-continuous at  $p = 0$ ). On the other hand, by applying the implicit function theorem to (22), we reach at the same conclusion for  $p \in (p_c, 1)$ . In Exercise 6.2, the reader is asked to prove that  $\theta$  is left-continuous at  $p = 1$ .

We will start by showing that, on the interval  $(p_c, 1)$ ,  $\theta(p)$  is strictly increasing.

PROPOSITION 6.1.  $\theta(p)$  is a strictly increasing function of  $p$  for  $p \in (p_c, 1)$ .

*Proof.* Let  $p_1 < p_2$  be two distinct values in  $(p_c, 1)$  and  $\theta_i = \theta(p_i)$ . One can check, from the definition of  $g(p, \theta)$ , that  $g(p_1, \theta) > g(p_2, \theta)$  for all  $\theta \in [0, 1]$ . In particular,  $\theta_1 = 1 - g(p_1, \theta_1) < 1 - g(p_2, \theta_1)$ ; i.e., at  $\theta_1$ , the graph of  $g(p_2, \theta)$  is below the graph of  $f(\theta)$ . The intersection of the two graphs, occurring by definition at  $\theta_2$ , must take place to the right of  $\theta_1$ . This proves the assertion.  $\square$

EXERCISE 6.1. *Show that  $\theta(p)$  is a nondecreasing function of  $p$ , for  $p \in [0, 1]$ .*

EXERCISE 6.2. *Using (22) and Proposition 6.1, show that  $\theta(p) \rightarrow \theta(1) = 1$  as  $p \rightarrow 1$  from the left.*

*Remark.* A nondecreasing function is differentiable at almost all points. In particular,  $\theta'(p) \geq 0$  at the points  $p$  where the derivative is well defined.

We now show that  $\lim_{p \rightarrow p_c^+} \theta(p) = 0$ . As pointed out earlier, this result leads to the conclusion that  $\theta(p)$  is continuous at  $p_c$ .

PROPOSITION 6.2.  $\theta(p)$  is continuous at  $p_c$ .

*Proof.* From Corollary 5.2,  $\theta(p_c) = 0$ . Also, from (14) and the monotonicity of  $\theta(p)$ , it is clear that  $\lim_{p \rightarrow p_c^-} \theta(p) = 0$ . So, to prove the continuity of  $\theta$  at  $p_c$  we just have to prove that  $\lim_{p \rightarrow p_c^+} \theta(p) = 0$ . From Proposition 6.1, we know that  $\theta$  is an increasing function in the interval  $(p_c, 1)$ . Since  $\theta(p)$  is bounded below by 0, it follows that the limit  $\lim_{p \rightarrow p_c^+} \theta(p)$  exists. Let's call it  $\theta_0$  and suppose, by contradiction, that  $\theta_0 > 0$ . Evaluating (22) at the point  $(p, \theta(p))$ ,  $p > p_c$ , and taking the limit on both sides of it, we get

$$1 - \theta_0 = 1 - (1 - p_c \theta_0)^{r-1},$$

implying that (22) has a nonzero solution at  $p_c$ , a contradiction with Theorem 5.1.  $\square$

**7.  $\beta = 1$  for Any  $r \geq 3$ .** Now we are in a position to define the exponent  $\beta$ , introduced informally in section 1. Since  $\theta(p) \rightarrow 0$  as  $p \rightarrow p_c^+$ , we would like to know at which rate  $\theta$  approaches zero. If there is an exponent  $\beta > 0$  such that the limit

$$\lim_{p \rightarrow p_c^+} \frac{\theta(p)}{(p - p_c)^\beta}$$

exists and is nonzero, then  $\beta$  is the *critical exponent* for  $\theta(p)$  (see section 4). In this section we will prove that this is the case. Moreover, we will show that  $\beta = 1$  for any  $r \geq 3$ .

PROPOSITION 7.1. On  $\mathcal{B}_r$ ,

$$\lim_{p \rightarrow p_c^+} \frac{\theta(p)}{(p - p_c)} = \frac{2}{p_c(1 - p_c)} \quad \forall r \geq 3.$$

*Proof.* The binomial expansion of the right side of (22) leads to

$$(25) \quad 1 - \theta = 1 - \left(\frac{\theta p}{p_c}\right) + \sum_{k=2}^{r-1} A_k \left(\frac{-\theta p}{p_c}\right)^k,$$

where

$$(26) \quad A_k \equiv \frac{1}{k!} (1 - p_c)(1 - 2p_c)(1 - 3p_c) \cdots (1 - [k - 1]p_c), \quad 2 \leq k \leq r - 1,$$

since

$$\begin{aligned} \binom{r-1}{k} &= \frac{1}{k! p_c^k} (1 - p_c)(1 - 2p_c)(1 - 3p_c) \cdots (1 - [k - 1]p_c) \\ &= \frac{A_k}{p_c^k} \quad \forall 2 \leq k \leq r - 1. \end{aligned}$$

From (25) we get

$$(27) \quad \theta \left(\frac{p}{p_c} - 1\right) = A_2 \theta^2 \left(\frac{p}{p_c}\right)^2 - A_3 \theta^3 \left(\frac{p}{p_c}\right)^3 + \cdots + (-1)^{r-1} A_{r-1} \theta^{r-1} \left(\frac{p}{p_c}\right)^{r-1}.$$

Since  $\theta(p) > 0$  for  $p > p_c$ , we divide (27) by its left-hand side to get

$$(28) \quad 1 = \left(\frac{p}{p_c} - 1\right)^{-1} \left[ A_2 \theta \left(\frac{p}{p_c}\right)^2 - A_3 \theta^2 \left(\frac{p}{p_c}\right)^3 + \cdots + (-1)^{r-1} A_{r-1} \theta^{r-2} \left(\frac{p}{p_c}\right)^{r-1} \right] \\ = \left(\frac{p_c \theta}{p - p_c}\right) \left[ A_2 \left(\frac{p}{p_c}\right)^2 - A_3 \theta \left(\frac{p}{p_c}\right)^3 + \cdots + (-1)^{r-1} A_{r-1} \theta^{r-3} \left(\frac{p}{p_c}\right)^{r-1} \right].$$

Observing that the function between brackets in identity (28) approaches  $A_2$  as  $p \rightarrow p_c^+$  and that  $A_2 = \frac{1}{2}(1 - p_c)$  (see definition (26)), we conclude from (28) that

$$\lim_{p \rightarrow p_c^+} \frac{\theta(p)}{(p - p_c)} = \frac{2}{p_c(1 - p_c)}. \quad \square$$

**COROLLARY 7.1.** *For  $r \geq 3$ , the critical exponent  $\beta$  equals 1.*

*Remark.* Hara and Slade [13, 14] proved the *triangle condition*  $\nabla(p_c) < \infty$ , where

$$\nabla(p) \equiv \sum_{x, y \in \mathbb{Z}^d} \tau_{0x}(p) \tau_{xy}(p) \tau_{y0}(p)$$

if  $d \geq 19$ . Barsky and Aizenman [2] proved that, for the bond percolation model on  $\mathbb{Z}^d$  under the triangle condition, the critical exponent  $\beta$  exists and is equal to 1. Both results together imply that  $\beta = 1$  for  $d \geq 19$  although it is believed that  $\beta = 1$  for  $d \geq 6$ . It also follows from both results that  $\theta(p)$  is continuous at  $p_c$  (see the remark just before section 6) if  $d \geq 19$ . It is an open question whether or not  $\theta(p)$  is continuous at  $p_c$ , for  $3 \leq d < 19$  (for  $d = 2$ , this result follows from [15]).

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